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A novel non-probabilistic approach using interval analysis for robust design optimization †

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Abstract

A technique for formulation of the objective and constraint functions with uncertainty plays a crucial role in robust design optimization. This paper presents the first application of interval methods for reformulating the robust optimization problem. Based on interval mathematics, the original real-valued objective and constraint functions are replaced with the interval-valued functions, which directly represent the upper and lower bounds of the new functions under uncertainty. The single objective function is converted into two objective functions for minimizing the mean value and the variation, and the constraint functions are reformulated with the acceptable robustness level, resulting in a bi-level mathematical model. Compared with other methods, this method is efficient and does not require presumed probability distribution of uncertain factors or gradient or continuous information of constraints. Two numerical examples are used to illustrate the validity and feasibility of the presented method.

***Keywords*:** Non-probabilistic; Interval analysis; Uncertainty analysis; Robust design optimization

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1. Introduction

Robust design optimization is a methodology to obtain an optimal design which is minimally sensitive to the uncertainties inherently present in the design process and parameters [1, 2]. To meet the need of higher product quality, engineers have increasingly applied optimization under uncertainty as an alternative to deterministic optimization. Robust design optimization is one of the representative methods under uncertainty. In this field, researchers have shown great enthusiasm in how to modify the optimization formulation for accounting for the uncertainties, which leads to two issues: objective robustness and constraints robustness. Objective robustness is typically achieved by simultaneously optimizing the mean performance and minimizing the variation of performance. Constraint robustness is to leave exact room for considering the variation of design variables by revising the formulation.

According to the characteristics of uncertainties and the way they have been treated, robust design optimization can be classified into two categories: probabilistic and non-probabilistic. Probabilistic methods have been a relatively mature and significant portion of the literature in this area, where the uncertainty is modeled into the optimization formulation using the theory of probability, and maintain design constraint satisfaction at an expected probability level. The common properties of the probabilistic method are that they require the probability distribution functions of uncertain factors as input [3-7]. The issue is that multidimensional integration has been involved in this field, and generally, a large amount of statistic information might be unknown or difficult to obtain. And recent research has shown that probabilistic results are very sensitive to the distribution data, which

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means that a slight error of the statistical data may lead to a great deviation of the results.

The non-probabilistic approaches do not need the presumed probabilistic information. However, these methods have some limitations, such as their objective or constraint functions must be differentiable with respect to the parameters or assume that the objective or constraint functions can be treated as linear with respect to the parameter variations, which might not hold for large variations [8-10]. References [11-13] analyze the mapping relationship between the parameter variation region and objective or constraint sensitivity region one by one, and obtain the variation quantity, then add the variation into the original formulation as a penalty for the robust design optimization, which does not need the gradient information of constraints. However, the analysis process is complicated and the optimal solution might be under- or over-conservative.

In view of the above-mentioned shortcomings, the aim of this paper is to develop a new robust optimization method to address some of deficiency. In recent years, the interval analysis method, which was first presented by Moor in the mid-1960s [14] and the linear interval equations and nonlinear interval equations have been resolved [15-17], has been developed to model the uncertainty in uncertain optimization problems, which only needs the upper and lower bounds of the uncertainties, and that is easier to obtain in actual engineering, not necessarily knowing the probability distributions or membership functions. This is the inspiration source and motivation of this paper.

This paper presents the first application of interval methods for reformulating the robust optimization mathematical model considering uncertainties. When the uncertainties are modeled as interval numbers, the related objective and constraint functions will also be interval functions. This paper combined interval extension of function algorithm and an order relation of interval number algorithm to evaluate the new changed upper and lower bounds of the objective and constraint functions under uncertainties, then convert them into equivalent deterministic form to obtain the optimum solution with any specified robustness and result in a bi-level mathematical model. The proposed method is capable of evaluating the robustness and achieving an optimal solution with any acceptable robustness levels, efficiently. Compared with other methods, our method is efficient, and does not require presumed probability distribution of uncertain factors or gradient or continuous information of uncertain factors; more information of the optimization results can be obtained and it is suitable for objective robustness and constraints robustness.

The remainder of this paper is presented as follows. In Section 2, the bi-level mathematical model based on interval analysis is developed. Section 3 illustrates our approach through two numerical examples, and concluding remarks are given in Section 4.

2. Robust design optimization problem formulation

In this section, firstly, a conventional robust optimization problem is shown and the relevant interval definition and terminology are introduced, followed by the interval mathematics. Then the reformulation of the objective function, inequality and equality constraint functions, based on interval mathematics, are developed. Finally, a bi-level mathematical model is constructed.

2.1 Interval mathematics

A general robust design optimization problem can be formulated as follows:

$$\begin{aligned} &\min f(\mathbf{X},\mathbf{P}) \\ &\text{subject to: } g_i(\mathbf{X},\mathbf{P}) \le 0, \quad i=1,2,...n \\ &\qquad\qquad h_j(\mathbf{X},\mathbf{P}) = 0, \quad j=1,2,...m \\ &\qquad\qquad \mathbf{X}_{\min} \le \mathbf{X} \le \mathbf{X}_{\max} \end{aligned} \tag{1}$$

Where $f(\mathbf{X},\mathbf{P})$is the objective function to be optimized, and $\mathbf{X}=[x_1,x_2,...x_k]$ is the vector of design variables which is to be assigned by the designer. $\mathbf{X}_{\min}$ and $\mathbf{X}_{\max}$ are the minimum and the maximum acceptable values for design variable vector $\mathbf{X}$. $g_i(\mathbf{X},\mathbf{P})$ is the ith inequality constraint function, and n is the number of inequality constraints. $h_j(\mathbf{X},\mathbf{P})$ is the jth equality constraint function, and m is the number of equality constraints. $\mathbf{P}=[p_1,p_2,...p_t]$ is the t-dimensional uncertain parameter vector, which is used to describe the uncertain effects. If the probabilistic distributions of all uncertain parameter variables are known, the robust design optimization problem can be solved by the method introduced in Refs [3-7]. However, in some cases, the probabilistic characteristics of the uncertain parameters are unknown, and the only known information is the interval of the uncertain parameters, namely [14]

$$\mathbf{P}^L \le \mathbf{P} \le \mathbf{P}^U \tag{2}$$

or the component form

$$p_i^L \le p_i \le p_i^U, \quad i = 1,2,...t \tag{3}$$

Where $\mathbf{P}^U=[p_1^U,p_2^U,...p_t^U]$ and $\mathbf{P}^L=[p_1^L,p_2^L,...p_t^L]$ are the upper and lower bounds vector of the uncertain-but-bounded parameter vector $\mathbf{P}=[p_1,p_2,...p_t]$, respectively. The relationship between the interval number and the real number can be written as:

$$\begin{aligned} &\mathbf{P} \in \mathbf{P}^I = [p_1^I, p_2^I, ..., p_t^I] = [\mathbf{P}^L, \mathbf{P}^U] or \\ &p_i \in p_i^I = [p_i^L, p_i^U], i = 1,2,...t \end{aligned} \tag{4}$$

Eq. (4) can be put into the more useful form as follows:

$$\mathbf{P}^I = [\mathbf{P}^c - \mathbf{P}^w, \mathbf{P}^c + \mathbf{P}^w] = \mathbf{P}^c + [-1,1]\mathbf{P}^w \tag{5}$$

Where $\mathbf{P}^c$ and $\mathbf{P}^w$ and denote the middle vector and the radius vector of $\mathbf{P}^I$, respectively. It follows that

$$\mathbf{P}^c = (\mathbf{P}^L + \mathbf{P}^U)/2 \tag{6}$$

$$\mathbf{P}^w = (\mathbf{P}^U - \mathbf{P}^L)/2 \tag{7}$$

Then the uncertain-but-bounded parameter vector P could be denoted as the following vector form:

$$\begin{aligned} &\mathbf{P} = \mathbf{P}^c + \delta\mathbf{P}, \delta\mathbf{P} \in [-1,1]\mathbf{P}^w \ or \\ &p_i = p_i^c + \delta p_i, \delta p_i \in [-1,1]p_i^w, i = 1,2,...t \end{aligned} \tag{8}$$

An interval function is an interval-value function of one or more interval arguments. In Eq. (1), assume that $F(\mathbf{X},\mathbf{P}^I)$is the interval value function of t-dimensional interval vector $\mathbf{P}^I$. Then the real function $f(\mathbf{X},\mathbf{P})$, which is the real function of t-dimensional real variables $\mathbf{P}$, satisfies the following properties:

$$f(\mathbf{X},\mathbf{P}) = F(\mathbf{X},\mathbf{P}^I) \tag{9}$$

F is known as the natural interval extension of f.

In Eq. (1), the real variables $\mathbf{P}=[p_1,p_2,...p_t]$ and the real arithmetic operations can be replaced with the corresponding interval variables $\mathbf{P}=[p_1,p_2,...p_t]$ and interval arithmetic operations, respectively, to obtain the natural interval extension functions $F(\mathbf{X},\mathbf{P}^I)$, $g_i'(\mathbf{X},\mathbf{P}^I)$ and $h_j'(\mathbf{X},\mathbf{P}^I)$of original real-valued objective and constraint functions $f(\mathbf{X},\mathbf{P})$, $g_i'(\mathbf{X},\mathbf{P})$ and $h_j'(\mathbf{X},\mathbf{P})$.

An optimization problem with interval numbers always needs to rank intervals to obtain the minimum or maximum one. The rank of interval numbers means that an interval number is better than another but not that one is larger than another. A nonlinear interval number programming called an order relation is often used to rank the intervals. Ishibuchi and Tanaka [16] used $\le_{mw}$to define an order relation between interval numbers A^I and B^I in the minimization problem as follows:

$$\begin{cases} A^I \le_{mw} B^I, & if \quad A^c \ge B^c and \quad A^w \ge B^w \\ A^I <_{mw} B^I, & if \quad A^I \le_{mw} B^I and \quad A^I \ne B^I \end{cases} \tag{10}$$

Eq. (10) means that interval B^I is better than A^I only if the midpoint and radius of B^I are both smaller than A^I.

2.2 Objective function formulation

Because the uncertainty factors are modeled as interval numbers, the real-valued objective function is replaced with the equivalent interval-valued objective function. The process of searching the minimum interval function value always keeps the relatively minor one through comparing between two interval functions until finding the minimum one. Based on the above-mentioned interval algorithm, the minimum interval value of the objective function under uncertainties means not only the smallest midpoint but also a smallest radius. Thus, the objective function with uncertainty in Eq. (1) can be transformed into a two-objective optimization problem as follows:

$$\begin{aligned} &\min f(\mathbf{X},\mathbf{P}) = \min F(\mathbf{X},\mathbf{P}^I) = \min(F^c, F^w) \\ &F^c = \frac{1}{2}(F^L(\mathbf{X},\mathbf{P}^I) + F^U(\mathbf{X},\mathbf{P}^I)) \\ &F^w = \frac{1}{2}(F^U(\mathbf{X},\mathbf{P}^I) - F^L(\mathbf{X},\mathbf{P}^I)) \end{aligned} \tag{11}$$

For a specific optimal solution $\mathbf{X}$, $F(\mathbf{X},\mathbf{P}^I)$ is an interval number, and its bounds $F^L(\mathbf{X},\mathbf{P}^I)$ and $F^U(\mathbf{X},\mathbf{P}^I)$ can be obtained:

$$F^L(\mathbf{X},\mathbf{P}^I) = \min_{P\in\Omega} f(\mathbf{X},\mathbf{P})$$
$$F^U(\mathbf{X},\mathbf{P}^I) = \max_{P\in\Omega} f(\mathbf{X},\mathbf{P}) \quad (12)$$
$$\mathbf{P} \in \Omega = \left\{\mathbf{P} \middle| \mathbf{P}^L \le \mathbf{P} \le \mathbf{P}^U\right\}$$

Through Eq. (12), the uncertain vector $\mathbf{P}^I$ can be eliminated, thus the two objective functions in Eq. (11) become deterministic.

The two objective functions in Eq. (11) have the practical significance of minimizing simultaneously the mean value and the deviation of the objective function caused by uncertainties, respectively. Through minimizing the variance of the objective function caused by the uncertainties, the optimal design can make the objective function insensitive to the perturbation of the uncertain factors. Therefore, objective robustness can be guaranteed. Note that in actual engineering, the designers are not always seeking the smallest variation of the objective function but the smallest objective average value in certain extent perturbation range. If the perturbation range Δf is presumed, the two objective functions in Eq. (11) can be converted into a single objective function and an inequality constraint function as follows:

$$\min f(\mathbf{X},\mathbf{P}) = \min(F^c)$$
$$subject \;\; to: F^w \le \Delta f \quad (13)$$

When the allowable variation range of objective function is known, Eq. (13) is an alternative formulation with practical significance in decreasing the cost of manufacturing.

2.3 Inequality constraint function formulation

In an engineering design problem, an optimal solution without considering uncertainties is usually located at or near the boundary of active constraints. A slight perturbation of the uncertainties may change the original constraint boundary, and the optimal solution evaluated from the conventional method may violate the new constructed constraints. We will take the ith constraint g_i to depict this situation as shown in Fig. 1. g_i' is the transformed new constraint of the original constraint g_i. The shadowed region is the added infeasible region by the variation of the constraint g_i. In Fig. 1(a), both optimum points $\boldsymbol{x}_{k\min}$ and $x_{k\max}$ obtained from the traditional method lie out of the feasible region; however, in another Fig. 1(b), $\boldsymbol{x}_{k\min}$ still satisfies the new constructed constraint with the variation caused by the uncertainties; unfortunately, $x_{k\max}$ violates the new constraint boundary. On the contrary, both optimal solutions in Fig. 1(c) lie in the feasible region.

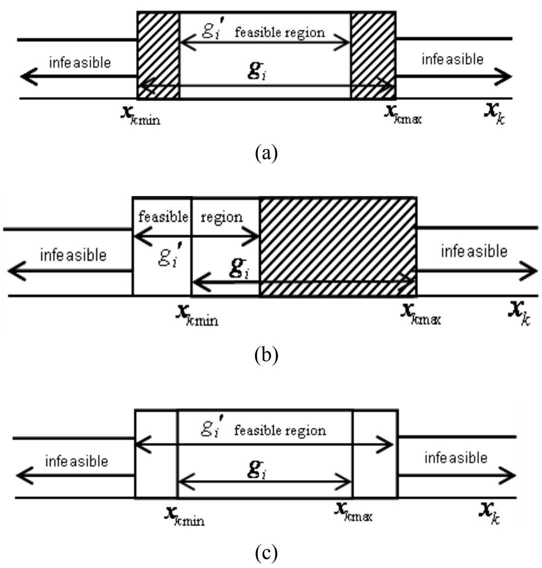

Fig. 1. The feasible robustness of the ith constraint g_i.

Constraint robustness is that the feasibility of the constraints is guaranteed even with the existence of the uncertainties. Because the uncertain factors are modeled as the interval numbers, the new constructed constraints will be interval functions. It is well known that, in robust design optimization, the constraint intervals only need to compare with zero; the part that is smaller than zero implies feasible. However, the part that is larger than zero represents the failure of robustness. References [16-17] proposed a construction method to compare the two intervals, which are called the possibility degree $P_{A^I \ge B^I}$ or $P_{B^I \ge A^I}$. When one of the intervals A^I or B^I degenerates into a real number ε, we can obtain the possibility degree between the interval-valued constraint g_i' and the real number, such as the degree $P_{g_i' \le \varepsilon}$, which can be written as follows:

$$P_{g_i' \le \varepsilon} = \begin{cases} 0 & g_i'^L \ge \varepsilon \\ (\varepsilon - g_i'^L)/(g_i'^U - g_i'^L) & g_i'^L \le 0 \le g_i'^U \\ 1 & g_i'^U \le \varepsilon \end{cases} \quad (14)$$

In the same way, $P_{g_i' \ge \varepsilon}$ can also be obtained:

$$P_{g_i' \ge \varepsilon} = \begin{cases} 1 & g_i'^L \ge \varepsilon \\ (g_i'^U - \varepsilon)/(g_i'^U - g_i'^L) & g_i'^L \le 0 \le g_i'^U \\ 0 & g_i'^U \le \varepsilon \end{cases} \quad (15)$$

It can be found when ε=0, $P_{g_i' \le 0}(\in [0,1])$ implies the constraint robustness called robustness index. When the robustness level does not satisfy the robustness requirement, we need to take some approaches to shift the constraint side such that the combinations of the uncertain variables for each constraint still result in a feasible design. The quantity of shift is determined by the robustness index calculated through Eq. (14). $P_{g_i' \le 0} = 1$ represents that the optimal solution obtained from the original optimization method completely satisfies the feasibility robustness requirement; hence the formulation of inequality constraints can maintain unchanged $g_i(\mathbf{X,P}) \le 0$. While $P_{g_i' \le 0} \le 1$ implies that the optimal solution evaluated from the original optimization method satisfies the new transformed constraints with the degree $P_{g_i' \le 0}$; obviously, $(1 - P_{g_i' \le 0})$ represents the degree that the optimal solution violates the new constructed constraint. The quantity of shift from the original constraint should be the part that does not satisfy the feasibility robustness requirement, which can be expressed as $2g_i'^w \times (1 - P_{g_i' \le 0})$, where 2 $g_i'^w$ is the interval width of the new constructed inequality constraint g_i', which can be calculated similar as F^w. Therefore the original inequality constraint is reformulated as follows:

$$g_i(\mathbf{X},\mathbf{P}) + 2(1 - P_{g_i' \le 0}) g_i'^w \le 0 \quad (16)$$

The shifted inequality constraints in Eq. (16) completely satisfy the robustness requirement. In the real world, the amount of shift needed for an inequality constraint can be related to the practical needs in the design. In some circumstances the design objective is of most concern and the constraints are allowed to be violated to a certain extent. A relatively lower robustness level can be selected, and in other situations the constraints are most important; then a relatively large robustness level should be specified. So a mathematical model with any satisfactory level needs to be formulated for the inequality constraints. Assume that φ_i is the specified robustness index of the *i*th inequality constraints by the decision maker. Besides, all the inequality constraints can be given the same φ, or completely different φ_i according to the preference of the decision maker and the practical needs. Therefore, the inequality constraints in Eq. (1) can be transformed into the following formulation with a specified robustness index:

$$P_{g_i' \le 0} \ge \varphi_i, g_i' = [g_i^L, g_i^U], i = 1,2,...n \quad (17)$$

Note that Eq. (16) is the collapsed form of Eq. (17) when the robustness index of all the constraints specified by the decision maker is equal to one $P_{g_i' \le 0} \ge 1$, so the inequality constraint can be summarized in one form as Eq. (17).

2.4 Equality constraint functions formulation

Reference [18] categorizes equality constraints into two types: 1) those that must be satisfied regardless of the uncertainty present, and 2) those that cannot be satisfied because of the uncertainty. The first type can be removed through substitution [19]. In this paper we put our emphasis on the second type. Similarly, as the inequality constraints, because the uncertain factors are modeled as interval numbers, the equality constraints are also interval functions. The difference between the inequality and equality constraints is that the former only need to focus on shifting the components larger than zero, and the latter one should pay attention to all the components that are not equal to zero. The reformulation process of the equality constraints is similar to the inequality constraints. The interval equality constraint function $h_j' = [h_j^L, h_j^U] = 0$ can be transformed into the following form:

$$h_j^L \le h_j'(\mathbf{X},\mathbf{P}) \le h_j^U \quad (18)$$

Eq. (18) can be converted into the following two inequality constraints:

$$\begin{cases} h_j'(\mathbf{X},\mathbf{P}) \le h_j^U \\ h_j'(\mathbf{X},\mathbf{P}) \ge h_j^L \end{cases} \quad (19)$$

Assuming that φ_j is the specified robustness index of the two inequality constraints, Eq. (19) can be changed into the following forms:

$$\begin{cases} P_{h_j' \le h_j^U} \ge \varphi_j \\ P_{h_j' \ge h_j^L} \ge \varphi_j \end{cases}, h_j' = [h_j^L, h_j^U], j = 1,2,...m \quad (20)$$

The possibility degrees $P_{h_j' \le h_j^U}$ and $P_{h_j' \ge h_j^L}$ can be calculated through Eqs. 14 and 15. Thus, the uncertain equality constraint has been converted into two deterministic inequality constraints.

2.5 Bi-level robust optimization model

Incorporating all of the above-mentioned reformulations, the conventional robust optimization model in Eq. (1) is transformed into the following bi-level mathematical model.

As shown in Fig. 2, the bi-level mathematical model of robust design optimization based on interval analysis consists of two steps. Firstly, optimize the objective function f subject to conventional constraints to get a solution x_0 in the space of x in upper-level subproblem, of which the variables are x; secondly, transfer x_0 to the lower-level subproblem to estimate the value of the middle vector and radius vector of the intervals($F^I, g_i'^I, h_j'^I$) in uncertainty space, of which the variables are $\mathbf{P}^I$; Then send back $F^c, F^w, g_i'^L, g_i'^U, h_j'^L, h_j'^U$ to the upper-level problem, and then the whole optimization continues until the upper-level problem converges to an optimum solution. It is notable that in a lower-level problem the aim is only to calculate the interval values and measure the robustness index for a particular solution x_0; therefore, x_0 is kept fixed in the lower-level subproblem.

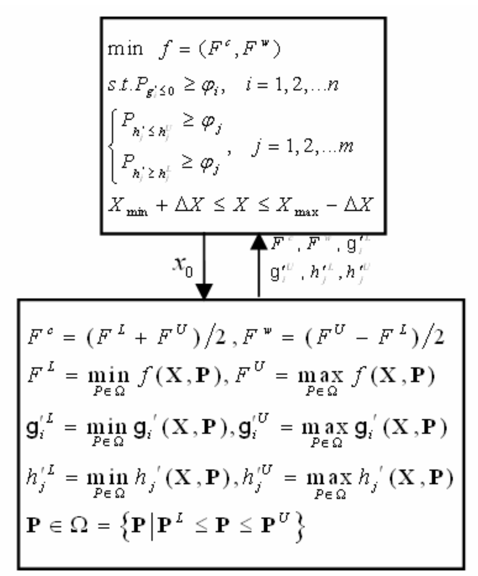

Fig. 2. Bi-level mathematical model of robust design optimization based on interval analysis.

3. Numerical examples

In this section the present method is applied into two examples to demonstrate the procedure. The first example is a mathematical problem. The second one is an engineering application of a welded beam, which is compared with three other methods. As we shall see, the results verify the above-mentioned advantage of the proposed approach.

3.1 A mathematical problem

The mathematical model is given by:

$$\begin{aligned} &\min f(\mathbf{X}, \mathbf{P}) = P_1(X_1 - 1.5)^2 + P_2(X_2 - 1)^2 + P_3 X_3 \\ &s.t.\ P_4 \ge P_1 X_1 + P_2 X_2 + P_3^2 X_3^2 + 1 \\ &\quad P_1 X_1^2 - P_2 X_2^2 + P_3 X_3^2 = P_5 \\ &\quad -1 \le X_1 \le 5, -3 \le X_2 \le 6, -2 \le X_3 \le 7, \end{aligned}$$

The problem involves three design variables $\mathbf{X}$=[x_1,x_2, x_3], five design parameters $\mathbf{P}$=[p_1,p_2,p_3,p_4,p_5]=[1.15,1,1.3,12.5,6.75], inequality constraint function and one equality constraint. Both the design variables and the design parameters have the uncertain factors. Δp_1=±0.15, Δp_2=Δp_3=±0.1, Δp_4=±2.5, Δp_5=±0.25, Δx_1=Δx_2=Δx_3=±0.1,

Firstly, the uncertain optimization problem is reformulated into the bi-level robust optimization model according to Fig. 2. The single objective function with uncertainty is converted into two-objective functions. Considering that the linear weighted algorithm is easy to implement, we adopt this method to integrate the multi-objective optimization. The value of the weighting factor is determined depending on the importance of objective value and robustness; in this example, we assume the two objective functions have the same preference and the objective functions have been normalized. The inequality and equality constraints are shifted into three inequality constraints, and all the constraints are specified the same robustness φ. The robustness index of the three inequality constraints is denoted by R1, R2 and R3, respectively. The optimal solutions are listed in Table 1.

As shown in Table 1, it can be found that with the increase of robustness index, the objective value is becoming worse, which means that the objective value and the constraint robustness are always conflicting, and they cannot be obtained meanwhile. A larger robustness index, which can guarantee the

Table 1. Optimal result under different robustness index φ.

φ	R1	R2	R3	the optimum	f
0.0	0.00	1.00	0.00	(1.64,1.08,1.70)	1.21
0.2	0.20	0.77	0.52	(1.59,1.10,1.92)	1.36
0.4	0.40	0.40	0.87	(1.52,1.18,2.14)	1.51
0.6	0.60	0.60	0.64	(1.34,1.54,2.33)	1.81
0.8	0.72	0.37	0.91	(0.28,1.37,2.63)	2.88
1	0.88	0.23	1.00	(0.60,1.59,2.73)	3.30

constraints are not violated in a larger possibility, always sacrifices a better objective value at great cost. Therefore, the decision maker needs to select an available tradeoff between the better optimum and the larger robustness. The results verify that our method can efficiently achieve an optimum with any specified robustness index.

3.2 A welded beam optimization

The well-known welded problem shown in Fig. 3 was originally formulated by Ragsdell and Phillips [20]. The optimal problem is used to demonstrate our robust optimization method based on interval analysis. The design variables and the design parameters are modified to have the uncertain factors.

The characteristics of the welded beam are as follows: the modulus of elasticity E is 30e6 psi (206.8 GPa), and the modulus of shear G is 120e6 psi (82.7 GPa). The allowable normal stressσ_d is 30000 psi (206.8 MPa) and the allowable shear stress τ_d is 13600 psi (93.77 MPa). The length L of the unwelded beam is 14 inches (35.56 cm), and the force F acting at the tip of the beam is 6000 1b (26.6 kN); the costs of weld and beam material c_1 and c_2 are \$0.1047/inch^3 (\$0.0064/cm^3), \$0.0481/inch^3 (\$0.003/cm^3), respectively.

The problem is that beam A is designed to support a force F acting at the tip of it, and which is welded to a rigid support member B. The objective of this optimal problem is to minimize the total cost of making such an assembly and the cost variation is smaller than 0.04($\Delta f \leq 0.04$) and completely satisfies all the constraints on the shear stress, normal stress, deflection, and buckling load on the beam with the uncertain effect. Through Fig. 3, it is not hard to define the four design variables: thickness of the beam (t), width of the beam (b), the weld (h), and length of the weld (l).

The original formulation of the welded beam assembly problem is shown directly as:

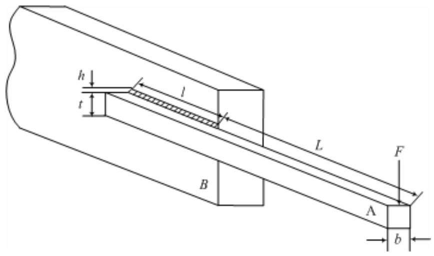

Fig. 3. A welded beam assembly.

Design variables: $\mathbf{X} = [x_1, x_2, x_3, x_4]^{\mathrm{T}} = [t, b, h, l]^{\mathrm{T}}$

Objective function:

$$\min f = (1+c_1)x_3^2 x_4 + c_2 x_1 x_2 (L + x_4)$$

Subjective to:

(1) Maximum shear stress constraint in weld

$$g_1(x) = \left[(\tau')^2 + 2\tau'\tau''\cos\theta + (\tau'')^2\right]^{1/2} - \tau_{\mathrm{d}} \leq 0$$

(2) Maximum normal stress constraint in beam

$$g_2(x) = 6FL/(x_2 x_1^2) - \sigma_{\mathrm{d}} \leq 0$$

(3) Bucking load constraint of the beam

$$g_3(x) = F - \frac{4.013\sqrt{EI\alpha}}{L^2}\left[1 - \frac{x_1}{2L}\sqrt{\frac{EI}{\alpha}}\right] \leq 0$$

(4) End deflection constraint of the beam

$$g_4(x) = 4FL^3/(Ex_1^3 x_2) - 0.25 \leq 0$$

(5) Other constraints

$$g_5(x) = x_3 - x_2 \leq 0$$

$$g_6(x) = 0.125 - x_3 \leq 0$$

$$0.1 \leq x_1 \leq 13, \quad 0.1 \leq x_2 \leq 2$$

$$0.1 \leq x_3 \leq 2, \quad 0.1 \leq x_4 \leq 10$$

L and c_1 have the uncertain factors, which are Δc_1=±0.05, ΔL=±0.25, respectively. Due to manufacturing errors x_1 and x_2 vary by±0.01. The detailed formulation of the problem is given in [21]. We solve the problem with four methods, traditional optimization method, the sensitivity region method [11], the maximum variation method [13], and the proposed method, which are denoted by M1, M2, M3 and M4 for convenient description. Table 2 shows the list of the optimum designs obtained from the four methods, and the list of constraint robustness is shown in Table 3, where R1, R2, R3, and R4 denote the robustness of the four inequality constraints, respectively.

Table 2. List of optimum designs.

	M1	M2	M3	M4
f	2.382	2.487	2.694	2.710
x_1	8.292	9.138	8.328	8.310
x_2	0.244	0.248	0.247	0.248
x_3	0.244	0.246	0.176	0.175
x_4	6.219	5.461	9.843	9.973

Table 3. List of robustness index.

	M1	M2	M3	M4
R1	0.505	0.573	0.990	1.000
R2	0.504	1.000	1.000	1.000
R3	1.000	1.000	1.000	1.000
R4	0.517	1.000	0.875	1.000

To compare the objective robustness in the four methods, 29 cases which are the randomly combinations of L and c_l in their variation bounds are simulated to evaluate the objective variation as shown in Fig. 4, where the dashed line is the maximum allowable variation value. Fig. 5 takes the fourth constraint as an example to show the variation of robustness index as a function of the interval width of the uncertain parameters, where the abscissa is the multiple of the interval width of the uncertainties, and the vertical coordinate is the first constraint robustness index. To compare the feasible space, we take the second inequality constraint as an example to show the original constraint curve and changed constraint curve in Fig. 5.

From the above results, it can be found that the traditional method has the best optimal solution, but the constraint robustness and the objective robustness are the worst. On the contrary, the optimal solution obtained from our method is the largest, but the constraint robustness and the objective robustness completely satisfy the requirement. Maximum variation method has similar results as our method, such as the optimum design and the objective robustness; however, the first and fourth constraint do not satisfy the demands. Sensitivity method has the larger constraint robustness, but the first constraint robustness was still violated, and the objective robustness is the worst.

As shown in Table 3, all four constraint robustness indexes of our method are larger than that of other three methods. This shows that the constraint robustness obtained by our method is the least sensitive to the uncertain factors.

As shown in Fig. 4, in all 29 cases the objective variation of M3 and our method is less than that of the

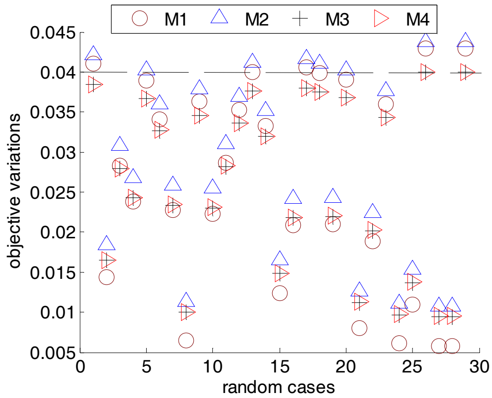

Fig. 4. Objective robustness comparison in four methods.

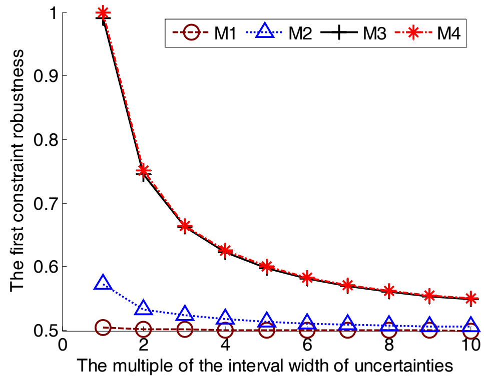

Fig. 5. Objective robustness comparison with respect to the interval width variation of uncertain parameters.

other two methods, while the M1 optimum violates the bounds in cases 1, 18, 26 and 29, and the M2 optimum violates the bounds in cases 1,14,18,19,26 and 29. This shows that the objective robustness obtained by our method is the least sensitive to the uncertain factors.

As shown in Fig. 5, from the fourth constraint, the general trend is that the constraint satisfaction probability decreases as the interval width increases of uncertain parameters. However, note that the proposed approach provides higher constraint robustness than other three approaches. This shows that the constraint robustness obtained by our method is guaranteed in a higher level with respect to the variation range of uncertain parameters.

As shown in Fig. 6, it can be found that as stated in section 2.2, by considering the uncertainties the feasible region is reduced and the presented method can estimate the change of constraints well.

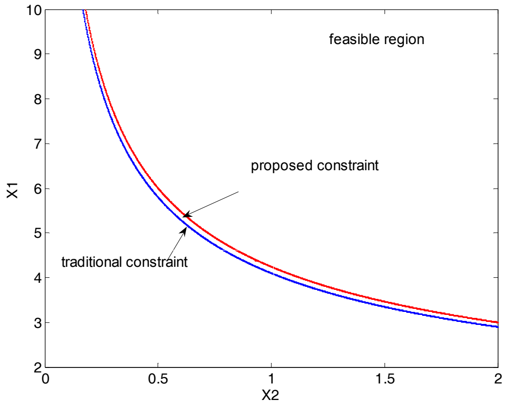

Fig. 6. Feasibility illustration of the second constraint.

4. Conclusions

A bi-level robust optimization model, combining interval extension of function algorithm and an order relation of interval number algorithm, is presented to investigate the effects of uncertainties on the objective and constraint functions for robust optimization.

The proposed method is applied to two numerical examples, and the first example verifies that our method can obtain an optimal solution with any specified robustness index. Through comparing with three other non-probabilistic methods, the second example confirms that the proposed method can guarantee the objective robustness and the constraint robustness. As interval mathematics is involved, only the upper and lower bounds of uncertainties are computed, so our computational process is more efficient. The result shows the advantages of our method are as follows: 1) It is straightforward, 2) It does not require presumed probability distribution of uncertain factors or gradient information of constraints or continuous requirement, 3) It requires less computational effort, and 4) is suitable for objective robustness and constraints robustness. So the presented method is more suitable for large engineering structures or substitutes stochastic optimization methods to solve some uncertain problems in which sufficient information on the uncertainty is unavailable.

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